A note on higher-order perturbative corrections to squirming speed in weakly viscoelastic fluids

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ABSTRACT

Many microorganisms swim in fluids with complex rheological properties. Although much is now understood about motion of these swimmers in Newtonian fluids, the understanding is still developing in non-Newtonian fluids—this understanding is crucial for various biomimetic and biomedical applications. Here we study a common model for microswimmers, the squirmer model, in two common viscoelastic fluid models, the Giesekus fluid model and fluids of differential type (grade three), at zero Reynolds number. Through this article we address a recent commentary that discussed suitable values of parameters in these models and pointed at higher-order viscoelastic effects on squirming motion.

1. Introduction

With ideas of minimally invasive surgery, targeted drug delivery, and other biomimetic applications [1–3], an understanding of motion of microswimmers in complex fluids has become imperative. Subsequently, many recent articles have focussed on motion of microswimmers in complex fluids (see reviews [4,5]). While biological fluids demonstrate many non-Newtonian fluid properties [6], one common property is viscoelasticity [7,8]. We consider this property in this article.

Viscoelastic fluids show both viscous and elastic properties, and retain memory of their flow history [9]. Recent experimental studies on biological swimmers [10–12] have addressed how an organism may change its swimming stroke as it “senses” the viscoelasticity of the fluid medium. Elastic stresses in the fluid can also directly contribute to changes in the swimming speed given a swimming stroke (see, for e.g., [13]). The present work is a theoretical study of swimmers in viscoelastic fluids. A model of microswimmers conducive to theoretical treatment is the squirmer model [14]. The model, developed by Lighthill [14] and Blake [15], consists of a rigid body that generates thrust due to the presence of (apparent) slip velocities on its surface. It has been used to understand various single and collective behaviours of microswimmers in Newtonian fluids [16]. In viscoelastic fluids, Zhu et al. [17] studied the motion of squirmers using numerical simulations and found that all squirmers—pushers, pullers and neutral swimmers—swim slower than in a Newtonian fluid for a wide range of values of the Weissenberg number (measure of viscoelasticity in the fluid). Later, De Corato et al. [18] using a theoretical approach (and the squirmer model), showed that in fact for very small values of the Deborah (Weissenberg) number not considered in the work of Zhu et al. [17] pusher swimmers swim faster, puller swimmers slower and neutral swimmers at the same speed as in a Newtonian fluid. We note that in these studies, as will be the case in the present study, the swimming speeds in viscoelastic and Newtonian fluids are compared for the same swimming stroke.

The work of De Corato et al. [18] used the second-order fluid model to study weakly viscoelastic effects on squirming motion. The use of the second order fluid model with parametric values as chosen by De Corato et al. [18] was critiqued by Christov and Jordan [19] who argued that the parametric values be chosen in accordance with thermodynamic constraints and recommended the use of other viscoelastic models which “better elucidate the transient effects of fluid viscoelasticity on a squirmer”. De Corato et al. [20] then showed that in fact using the Giesekus model to study weakly viscoelastic effects, to $O(De)$, validates their results [18] that were obtained using the second-order fluid model. The motivation for this work in large part is due to this discussion; here we study the squirming motion to higher orders in Deborah number both in the Giesekus fluid and in fluids of differential type. We find that unlike in a second-order fluid that obeys thermodynamic constraints, weak viscoelastic contributions to the squirming speed are non-zero in a fluid of grade three (third-order fluid) obeying thermodynamic constraints. These contributions are qualitatively different to those obtained due to viscoelasticity as modelled by the Giesekus fluid.

In the following, we briefly discuss the squirmer model and the second-order fluid model with the points of contention, and then present our results.

2. Theoretical framework

2.1. The squirmer model

The spherical squirmer model consists of a sphere with prescribed axisymmetric surface velocities (surface velocities may be thought of as originating from surface distortions in biological microswimmers like...
which generate thrust forces to propel the swimmer [14,15]. We consider only tangential surface velocities on the swimmer (the swimmer maintains its shape) so that the surface velocity $u^s = u^s_\text{dp}$, where $u^s_\text{dp}$ can be expressed as
\begin{equation}
    u^s_\text{dp} = \sum_{m=1}^{\infty} B_m V_m(\theta),
\end{equation}
using $V_m(\theta) = (-2/((l+1)))P_l^1(\cos \theta); \ P_l^1(\cos \theta)$ are associated Legendre polynomials of the first kind, and $\theta$ is the polar angle measured from the axis of symmetry [15]. The coefficients $B_m$ are generally referred to as squirming modes. In Newtonian fluids, the swimming speed of the squirm is due to just the first mode, $U_n = 2/3 B_1$, and the second mode $B_2$ gives the stresses due to the squirm [21]. As velocities due to higher modes decay faster than due to the first two modes (in fact, $B_2$ gives the slowest decaying spatial contribution to the flow field), and since higher modes do not contribute to the swimming speed, in Newtonian fluids, often only the first two modes are considered, i.e., $B_m = 0$ for $n > 3$. For the purpose of this study, in accordance with the bulk of literature in the field [16], we too consider only the first two modes. At this point we feel it is important to note that in general considering only the first two modes in complex fluids may be problematic as shown in the recent works of Datt et al. [22,23]. The interested reader may refer to the description of non-axisymmetric squirming modes in Newtonian fluids by Pak and Lauga [24].

When the ratio $\beta = B_2/B_1$ is negative, the squirm generates thrust from its rear end, like the bacterium *E. coli*; when $\beta > 0$ the thrust is generated from the front end, as in the “breaststroking” alga Chlamydomonas. When $\beta = 0$, the thrust and drag centres coincide, and flow field around the swimmer is due to a potential dipole. The three types of squirmers are called pushers, pullers, and neutral swimmers, respectively [16].

### 2.2. The second-order fluid model

The deviatoric stress in an incompressible second-order fluid is given by
\begin{equation}
    \tau = \eta A_1 + \alpha_1 A_2 + \alpha_2 A_1^2,
\end{equation}
where
\begin{equation}
    A_1 = L + L^T,
\end{equation}
\begin{equation}
    A_2 = \frac{D A_{\text{visc}}}{\text{dt}} + L^T A_{\text{visc},1} + A_{\text{visc},1} L,
\end{equation}
with $L^T = V u$ and $D/\text{dt}$ denoting the material derivative [25,26]. Here $\eta$ is the shear viscosity and $\alpha_1$ and $\alpha_2$ are material moduli. The second-order fluid model may be obtained as a second-order approximation to simple fluids with a particular kind of fading memory in slow flows [25,27,28]. The zeroth-order approximation gives the constitutive equation for the incompressible ideal fluid, whereas the first-order approximation results in the constitutive equation of the incompressible Newtonian fluid (the Navier–Stokes fluid) [25,27,28]. The third-order fluid equation is discussed later in the article. The second-order model has been used to study the first effects of viscoelasticity on the motion of both passive and active particles (see for e.g., [29–31]). However, there has been much discussion on the permissible values of $\alpha_1$ and $\alpha_2$ in the model. Dunn and Fosdick [32] have shown that considering (2.2) as exact, the model is consistent with thermodynamics when
\begin{align}
    \eta &\geq 0, \\
    \alpha_1 &\geq 0, \\
    \alpha_1 + \alpha_2 &\geq 0.
\end{align}

However, often these constraints, citing experimental investigations (incorrectly, according to Dunn and Rajagopal [25]), are not strictly adhered to. In particular, $\alpha_1$, which corresponds to the first normal stress difference coefficient, is generally taken to be negative [25].

### 2.3. The reciprocal theorem

The reciprocal theorem of low Reynolds number hydrodynamics [33] can be used to calculate the first effects of the fluid rheology on the swimming speed of microswimmers [34]. The details of the reciprocal theorem for the specific case of squirmers in viscoelastic fluids may be found, among others, in the works of Lauga [35], De Corato et al. [18] and Datt et al. [23].

Consider a weakly non-linear fluid of the form [34]
\begin{equation}
    \tau = \eta \dot{\gamma} + \epsilon \Sigma[u],
\end{equation}
where $\dot{\gamma}$ is the deviatoric stress, $\eta$ is the shear viscosity, and $\dot{\gamma}$ is the strain rate tensor so that the first term on the right hand side in (2.8) gives the Newtonian contribution. Here $\epsilon$ is the small parameter that quantifies the deviation from the Newtonian behaviour and $\Sigma$ gives the non-Newtonian contribution. The translational velocity of a squirm of radius $a$ in such a fluid is, obtained by using the reciprocal theorem,
\begin{equation}
    U = \frac{1}{4\pi a^2} \int_S u^s dS - \frac{1}{8\pi a} \int_V \chi \cdot (1 + \frac{a^2}{6} V^2) \nabla \text{GdV},
\end{equation}
where $\nabla = (1/r)(I + \text{rr}/r^2)$ is the Oseen tensor, and $S$ denotes the surface of the swimmer, and $V$, the fluid volume [23].

### 3. Results and discussion

De Corato et al. [18] studied the motion of a squirm in a second-order fluid. Considering only small deviations from Newtonian behaviour, they expanded all flow quantities in the small parameter $\epsilon = De$, where Deborah number $De = -\alpha_1 B_1 / (\eta a)$ is a measure of the relaxation time scale of the fluid to the characteristic time scale of the flow (note that for steady surface slip velocity squirmers, the Deborah and Weissenberg numbers are equivalent [36]). De Corato et al. [18] assumed $\alpha_1 < 0$, in contradiction with the thermodynamic stability criterion as pointed out by Christov and Jordan [19]. The thermodynamic constraint $\alpha_1 + \alpha_2 = 0$ was also relaxed. De Corato et al. [18] found that the perturbation calculations predicted that pushers swim faster, pullers slower and neutral swimmers at the same speed as in Newtonian fluids, provided that the swimming gait remains unchanged between the viscoelastic and Newtonian fluids. Their numerical simulations in a Giesekus fluid found the analytical results to hold up to $De = 0.02$ [18]. It was commented that the deviation of the results due to theoretical calculations from those due to numerical simulations at larger $De$ was because of higher order viscoelastic effects that were neglected in the analytical results for which only $O(De)$ corrections were analysed [18].

The critique of the work of De Corato et al. [18] by Christov and Jordan [19] was focussed on the former not respecting the thermodynamic constraints of the second-order fluid model. In particular, Christov and Jordan [19] remarked that since $\alpha_1 + \alpha_2$ should be equal to zero, most corrections to flow quantities (but pressure) including the swimming speed of the squirm will be zero, since all these corrections are proportional to the sum $\alpha_1 + \alpha_2$. Citing [32], Christov and Jordan [19] also pointed out that for $\alpha_1 < 0$ a steady solution to the problem should not be expected. Finally, Christov and Jordan [19] suggested calculating corrections to the swimming motion with the thermodynamic constraints (meaning going to higher powers in $De$ for any non-zero contributions) or using a different viscoelastic model, such as the upper-convected Maxwell model.

De Corato et al. [20] then showed that even with using a more involved model like the Giesekus fluid model (which reduces to the upper-convected Maxwell model for a choice of a model parameter) one obtains equations identical to the second-order fluid in the limit of small $De$ at $O(De)$. Further, for its permissible values, the Giesekus fluid gives identical results to those from the second-order fluid as used by De Corato et al. [18]. In fact, they maintain that the second-order fluid model should be seen as an approximation to more complex viscoelastic models in slow and nearly steady flows (and therefore (2.2) not be seen
as exact). Perhaps, in order to avoid any confusion, one may restrict the use of the term “second-order fluid model” only when it is treated as an exact model obeying the thermodynamic constraints; where a slow and nearly steady flow approximation is used one cannot start with a more involved model and reduce it to simpler constitutive equations at each order in the perturbation series in De. Below we use this terminology and study the squirmer in a Giesekus fluid and in fluids of grade n (the second-order fluid is a fluid of grade two) and calculate the corrections to the swimming speed in these fluids to higher orders in De.

3.1. Giesekus fluid

The polymeric stress in an incompressible Giesekus fluid is given as

\[ \tau_p + \lambda \frac{\partial}{\partial t} \frac{\lambda}{\eta_p} \tau_p + \tau_p = \eta_p \dot{\gamma}, \]

where the mobility factor \( a_m \) must take values between 0 and 1/2 [17,37]. The total deviatoric stress in the fluid is \( \tau = \tau_p + \tau_r \) where \( \tau_r = \eta_r \dot{\gamma} \) is the contribution from the Newtonian solvent. The total viscosity in the fluid \( \eta = \eta_r + \eta_p \). The Giesekus model, derived from molecular ideas [38,39], has been successfully used to model experimental results, see, for example, [40–42]. Here we consider the case when \( \zeta = \eta_r/\eta = 0 \); when \( \zeta = 0 \) and \( a_m = 0 \), (3.1) reduces to the upper-convedt Maxwell fluid model [37].

We non-dimensionalise equations by scaling lengths by the squirmer radius \( a \), velocities with the first squirming mode \( B_1 \), and stresses with \( \eta B_1/a \), and obtain the dimensionless constitutive equation

\[ \tau^* + De \tau^* + a_m De \tau^* \cdot \tau^* = \dot{\gamma}^*, \]

where the Deborah number \( De = \lambda B_1/a \). Henceforth, we drop the stars for convenience. We expand all flow quantities in a regular perturbation expansion in \( De \), and use standard methods to calculate the flow fields in Stokes flow [33] obtain the swimming speed of the squirmer, up to \( O(De^3) \),

\[ U = \frac{2}{3} + \frac{2}{15} \beta (-1 + a_m) De + \beta^2 \left( \frac{-20568 - 98136 a_m + 65266 a_m^2}{45045} + 84 (-193 + 176 a_m (-3 + 2 a_m)) \right) De^2 \]

\[ + \frac{\beta}{482431950} \left( 170 (3005646 + a_m (6190100 + 3 a_m (-10014053 + 4815243 a_m))) \right) De^3. \]

At this point, examining Eq. (3.3) for specific values of \( \beta \) and \( a_m \) becomes instructive; we choose \( \beta = -1 \) for pushers, 0 for neutral squirmers, and 1 for puller-type squirmers and \( a_m = 0.2 \). These values correspond to the values used in the work of De Corato et al. [18]. From (3.3) we find, for pushers,

\[ \frac{U}{U_N} = 1 + 0.16 De - 2.05 De^2 - 2.62 De^3, \]

for pullers,

\[ \frac{U}{U_N} = 1 - 0.16 De - 2.05 De^2 + 2.62 De^3, \]

and for neutral squirmers,

\[ \frac{U}{U_N} = 1 - 0.80 De^2. \]

The swimming speeds in (3.4), (3.5), and (3.6) are plotted in Fig. 1 along with their respective Padé approximations \( P_2(De) \). When corrections up to only \( O(De) \) are considered, we note that pushers swim faster, pullers slower and neutral swimmers at the same speed as in a Newtonian fluid; this is shown in the work of De Corato et al. [18]. With Padé approximations of terms up to \( O(De^3) \), we note that all the squirmers swim slower than in a Newtonian fluid (except for very small values of \( De \)) as found in the numerical work of Zhu et al. [17]. Clearly, the inclusion of higher order terms changes the theoretical predictions significantly.

One may calculate terms to even higher-order in the expansion to predict results for larger values of \( De \). This is done by Housiadas and Tanner [44], up to \( O(De^8) \), for steady sedimentation of a passive sphere in a viscoelastic fluid. Housiadas and Tanner [44] also quantify when the results from the series should not be considered (using positive definiteness of the conformation tensor). Sazade et al. [45] and Elfing and Lauga [5] also performed a higher-order perturbation analysis, using techniques to improve the convergence properties of the series, for the swimming speed of a two-dimensional swimming sphere where the small parameter was the amplitude of the waves on the sheet. We have not pursued these endeavours here, for the motivation for this study was to see the differences between the different viscoelastic models considering only the first few terms.

The results in the foregoing were obtained using the Giesekus model for viscoelasticity. They would remain qualitatively the same if one were to use the upper-convedt Maxwell model. But what happens to a squirmer in a fluid of grade \( n \), when the fluid is “regarded as a fluid in its own right, not necessarily an approximation to any other one” [28]?

3.2. A fluid of grade three

Consider an incompressible fluid of grade three [46]:

\[ \tau = \eta A_1 + a_1 A_2 + a_2 A_2 + \frac{\beta_1 A_1 + \beta_2 A_2 + \beta_3 A_2}{2} \]

\[ + \beta_3 \left[ \frac{\eta (tr A_1)}{2} + A_1 \right], \]

where \( \eta, a_1, a_2, \beta_1, \beta_2, \) and \( \beta_3 \) are material moduli. The equation is dimensional. For a list of works using fluids of grade three, see [47]. Thermodynamics stipulates [46] that

\[ \eta \geq 0, \quad a_1 \geq 0, \quad \beta_1 \geq 0, \quad \beta_2 \geq 0, \quad \beta_3 \geq 0. \]

Fig. 1. Swimming speeds in the Giesekus fluid as a function of De. The solid lines include corrections up to \( O(De^3) \). The dashed lines are Padé approximations to the series for the speeds in the text. The dotted lines include only \( O(De) \) corrections. The addition of the higher order modes decreases the speeds of the squirmers. As seen here (dashed lines), all squirmers at large values of De swim slower than in a Newtonian fluid, as found in the numerical work of Zhu et al. [17].
altogether differently to some test of a different kind”. At higher $De$, all squirmers swim faster in fluids of grade three than in a Newtonian fluid, when in Giesekus fluids they would swim slower.

4. Conclusion

We calculated the higher order corrections to the swimming speeds in two viscoelastic fluids—the Giesekus fluid and the fluid of grade three. The higher order corrections significantly add to the results of $O(De)$; even at relatively small values of $De$, the corrections lead to qualitatively different speeds. This again raises the question about the range of values of $De$ at which the expansion can accurately predict results (also see [23]). Importantly, we observe that the two fluids, the Giesekus fluid and the fluid of grade three, predict qualitatively different swimming speeds for the squirmers. Clearly, the answer to what viscoelastic model to use depends on what all we wish to model—in this, we are guided by experiments.

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