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A note on higher-order perturbative corrections to squirming speed in weakly viscoelastic fluids

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A B S T R A C T

Many microorganisms swim in fluids with complex rheological properties. Although much is now understood about motion of these swimmers in Newtonian fluids, the understanding is still developing in non-Newtonian fluids—this understanding is crucial for various biomimetic and biomedical applications. Here we study a common model for microswimmers, the squirmer model, in two common viscoelastic fluid models, the Giesekus fluid model and fluids of differential type (grade three), at zero Reynolds number. Through this article we address a recent commentary that discussed suitable values of parameters in these models and pointed at higher-order viscoelastic effects on squirming motion.

1. Introduction

With ideas of minimally invasive surgery, targeted drug delivery, and other biomimetic applications [1–3], an understanding of motion of microswimmers in complex fluids has become imperative. Subsequently, many recent articles have focussed on motion of microswimmers in complex fluids (see reviews [4,5]). While biological fluids demonstrate many non-Newtonian fluid properties [6], one common property is viscoelasticity [7,8]. We consider this property in this article.

Viscoelastic fluids show both viscous and elastic properties, and retain memory of their flow history [9]. Recent experimental studies on biological swimmers [10–12] have addressed how an organism may change its swimming stroke as it "senses" the viscoelasticity of the fluid medium. Elastic stresses in the fluid can also directly contribute to changes in the swimming speed given a swimming stroke (see, for e.g., [13]). The present work is a theoretical study of swimmers in viscoelastic fluids. A model of microswimmers conducive to theoretical treatment is the squirmer model [14]. The model, developed by Lighthill [14] and Blake [15], consists of a rigid body that generates thrust due to the presence of (apparent) slip velocities on its surface. It has been used to understand various single and collective behaviours of microswimmers in Newtonian fluids [16]. In viscoelastic fluids, Zhu et al. [17] studied the motion of squirmers using numerical simulations and found that all squirmers—pushers, pullers and neutral swimmers—swim slower than in a Newtonian fluid for a wide range of values of the Weissenberg number (measure of viscoelasticity in the fluid). Later, De Corato et al. [18] using a theoretical approach (and the squirmer model), showed that in fact for very small values of the Deborah (Weissenberg) number not considered in the work of Zhu et al. [17] pusher swimmers swim faster, puller swimmers slower and neutral swimmers at the same speed as in a Newtonian fluid. We note that in these studies, as will be the case in the present study, the swimming speeds in viscoelastic and Newtonian fluids are compared for the same swimming stroke.

The work of De Corato et al. [18] used the second-order fluid model to study weakly viscoelastic effects on squirming motion. The use of the second order fluid model with parametric values as chosen by De Corato et al. [18] was critiqued by Christov and Jordan [19] who argued that the parametric values be chosen in accordance with thermodynamic constraints and recommended the use of other viscoelastic models which "better elucidate the transient effects of fluid viscoelasticity on a squirmer". De Corato et al. [20] then showed that in fact using the Giesekus model to study weakly viscoelastic effects, to $\mathcal{O}(De)$, validates their results [18] that were obtained using the second-order fluid model. The motivation for this work in large part is due to this discussion; here we study the squirming motion to higher orders in Deborah number both in the Giesekus fluid and in fluids of differential type. We find that unlike in a second-order fluid that obeys thermodynamic constraints, weak viscoelastic contributions to the squirming speed are non-zero in a fluid of grade three (third-order fluid) obeying thermodynamic constraints. These contributions are qualitatively different to those obtained due to viscoelasticity as modelled by the Giesekus fluid.

In the following, we briefly discuss the squirmer model and the second-order fluid model with the points of contention, and then present our results.

2. Theoretical framework

2.1. The squirmer model

The spherical squirmer model consists of a sphere with prescribed axisymmteric surface velocities (surface velocities may be thought of as originating from surface distortions in biological microswimmers like

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Opalina) which generate thrust forces to propel the swimmer [14,15]. We consider only tangential surface velocities on the swimmer (the swimmer maintains its shape) so that the surface velocity $\mathbf{u}^S = u_\theta^S \mathbf{e}_\theta$, where u_θ^S can be expressed as

$$u_\theta^S = \Sigma_{l=1}^{\infty} B_l V_l(\theta), \tag{2.1}$$

using $V_l(\theta) = -(2/(l(l+1)))P_l^1(\cos\theta)$; $P_l^1(\cos\theta)$ are associated Legendre polynomials of the first kind, and θ is the polar angle measured from the axis of symmetry [15]. The coefficients B_l are generally referred to as squirming modes. In Newtonian fluids, the swimming speed of the squirmer is due to just the first mode, $U_N = 2/3B_1$, and the second mode B_2 gives the stresslet due to the squirmer [21]. As velocities due to higher modes decay faster than due to the first two modes (in fact, B_2 gives the slowest decaying spatial contribution to the flow field), and since higher modes do not contribute to the swimming speed, in Newtonian fluids, often only the first two modes are considered, i.e., $B_n = 0$ for $n > 3$. For the purpose of this study, in accordance with the bulk of literature in the field [16], we too consider only the first two modes. At this point we feel it is important to note that in general considering only the first two modes in complex fluids may be problematic as shown in the recent works of Datt et al. [22,23]. The interested reader may refer to the description of non-axisymmetric squirming modes in Newtonian fluids by Pak and Lauga [24].

When the ratio $\beta = B_2/B_1$ is negative, the squirmer generates thrust from its rear end, like the bacterium *E. coli.*; when $\beta > 0$ the thrust is generated from the front end, as in the "breaststroking" algae *Chlamydomonas*. When $\beta = 0$, the thrust and drag centres coincide, and flow field around the swimmer is due to a potential dipole. The three types of squirmers are called pushers, pullers, and neutral swimmers, respectively [16].

2.2. The second-order fluid model

The deviatoric stress in an incompressible second-order fluid is given by

$$\boldsymbol{\tau} = \eta \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \tag{2.2}$$

where

$$\mathbf{A}_1 \equiv \mathbf{L} + \mathbf{L}^T, \tag{2.3}$$

$$\mathbf{A}_n \equiv \frac{\mathrm{D}\mathbf{A}_{n-1}}{\mathrm{D}t} + \mathbf{L}^T \mathbf{A}_{n-1} + \mathbf{A}_{n-1}\mathbf{L}, \tag{2.4}$$

with $\mathbf{L}^T = \nabla \mathbf{u}$, and $\mathrm{D}/\mathrm{D}t$ denoting the material derivative [25,26]. Here η is the shear viscosity and α_1 and α_2 are material moduli. The second-order fluid model may be obtained as a second-order approximation to simple fluids with a particular kind of fading memory in *slow* flows [25,27,28]. The zeroth-order approximation gives the constitutive equation for the incompressible ideal fluid, whereas the first-order approximation results in the constitutive equation of the incompressible Newtonian fluid (the Navier–Stokes fluid) [25,27,28]. The third-order fluid equation is discussed later in the article. The second-order model has been used to study the first effects of viscoelasticity on the motion of both passive and active particles (see for e.g., [29–31]). However, there has been much discussion on the permissible values of α_1 and α_2 in the model. Dunn and Fosdick [32] have shown that considering (2.2) as exact, the model is consistent with thermodynamics when

$$\eta \geq 0, \tag{2.5}$$

$$\alpha_1 \geq 0, \tag{2.6}$$

$$\alpha_1 + \alpha_2 = 0. \tag{2.7}$$

However, often these constraints, citing experimental investigations (incorrectly, according to Dunn and Rajagopal [25]), are not strictly adhered to. In particular, α_1, which corresponds to the first normal stress difference coefficient, is generally taken to be negative [25].

2.3. The reciprocal theorem

The reciprocal theorem of low Reynolds number hydrodynamics [33] can be used to calculate the first effects of the fluid rheology on the swimming speed of microswimmers [34]. The details of the reciprocal theorem for the specific case of squirmers in viscoelastic fluids may be found, among others, in the works of Lauga [35], De Corato et al. [18] and Datt et al. [23].

Consider a weakly non-linear fluid of the form [34]

$$\boldsymbol{\tau} = \eta \dot{\boldsymbol{\gamma}} + \varepsilon \boldsymbol{\Sigma}[\boldsymbol{u}], \tag{2.8}$$

where $\boldsymbol{\tau}$ is the deviatoric stress, η is the shear viscosity, and $\dot{\boldsymbol{\gamma}}$ is the strain rate tensor so that the first term on the right hand side in (2.8) gives the Newtonian contribution. Here ε is the small parameter that quantifies the deviation from the Newtonian behaviour and $\boldsymbol{\Sigma}$ gives the non-Newtonian contribution. The translational velocity of a squirmer of radius a in such a fluid is, obtained by using the reciprocal theorem,

$$\mathbf{U} = -\frac{1}{4\pi a^2}\int_S \mathbf{u}^S \mathrm{d}S - \varepsilon \frac{1}{8\pi\eta}\int_V \boldsymbol{\Sigma} : \left(1 + \frac{a^2}{6}\nabla^2\right)\nabla \mathbf{G}\mathrm{d}V, \tag{2.9}$$

where $\mathbf{G} = (1/r)\left(\mathbf{I} + \mathbf{r}\mathbf{r}/r^2\right)$ is the Oseen tensor, and S denotes the surface of the swimmer, and V, the fluid volume [23].

3. Results and discussion

De Corato et al. [18] studied the motion of a squirmer in a second-order fluid. Considering only small deviations from Newtonian behaviour, they expanded all flow quantities in the small parameter $\varepsilon = De$, where Deborah number $De = -\alpha_1 B_1/(\eta a)$ is a measure of the relaxation time scale of the fluid to the characteristic time scale of the flow (note that for steady surface slip velocity squirmers, the Deborah and Weissenberg numbers are equivalent [36]). De Corato et al. [18] assumed $\alpha_1 < 0$, in contradiction with the thermodynamic stability criterion as pointed out by Christov and Jordan [19]. The thermodynamic constraint $\alpha_1 + \alpha_2 = 0$ was also relaxed. De Corato et al. [18] found that the perturbation calculations predicted that pushers swim faster, pullers slower and neutral swimmers at the same speed as in Newtonian fluids, provided that the swimming gait remains unchanged between the viscoelastic and Newtonian fluids. Their numerical simulations in a Giesekus fluid found the analytical results to hold up to $De \approx 0.02$ [18]. It was commented that the deviation of the results due to theoretical calculations from those due to numerical simulations at larger De was because of higher order viscoelastic effects that were neglected in the analytical results for which only $\mathcal{O}(De)$ corrections were analysed [18].

The critique of the work of De Corato et al. [18] by Christov and Jordan [19] was focussed on the former not respecting the thermodynamic constraints of the second-order fluid model. In particular, Christov and Jordan [19] remarked that since $\alpha_1 + \alpha_2$ should be equal to zero, most corrections to flow quantities (but pressure) including the swimming speed of the squirmer will be zero, since all these corrections are proportional to the sum $\alpha_1 + \alpha_2$. Citing [32], Christov and Jordan [19] also pointed out that for $\alpha_1 < 0$ a steady solution to the problem should not be expected. Finally, Christov and Jordan [19] suggested calculating corrections to the swimming motion with the thermodynamic constraints (meaning going to higher powers in De for any non-zero contributions) or using a different viscoelastic model, such as the upper-convected Maxwell model.

De Corato et al. [20] then showed that even with using a more involved model like the Giesekus fluid model (which reduces to the upper-convected Maxwell model for a choice of a model parameter) one obtains equations identical to the second-order fluid in the limit of small De at $\mathcal{O}(De)$. Further, for its permissible values, the Giesekus fluid gives identical results to those from the second-order fluid as used by De Corato et al. [18]. In fact, they maintain that the second-order fluid model should be seen as an approximation to more complex viscoelastic models in slow and nearly steady flows (and therefore (2.2) not be seen

as exact). Perhaps, in order to avoid any confusion, one may restrict the use of the term "second-order fluid model" only when it is treated as an exact model obeying the thermodynamic constraints; where a slow and nearly steady flow approximation is used one can start with a more involved model and reduce it to simpler constitutive equations at each order in the perturbation series in *De*. Below we use this terminology and study the squirmer in a Giesekus fluid and in fluids of grade *n* (the second-order fluid is a fluid of grade two) and calculate the corrections to the swimming speed in these fluids to higher orders in *De*.

3.1. Giesekus fluid

The polymeric stress in an incompressible Giesekus fluid is given as [37]

$$\boldsymbol{\tau}_p + \lambda \overset{\nabla}{\boldsymbol{\tau}}_p + \alpha_m \frac{\lambda}{\eta_p} \boldsymbol{\tau}_p \cdot \boldsymbol{\tau}_p = \eta_p \dot{\boldsymbol{\gamma}}, \tag{3.1}$$

where the mobility factor α_m must take values between 0 and 1/2 [17,37]. The total deviatoric stress in the fluid is $\boldsymbol{\tau} = \boldsymbol{\tau}_s + \boldsymbol{\tau}_p$ where $\boldsymbol{\tau}_s = \eta_s \dot{\boldsymbol{\gamma}}$ is the contribution from the Newtonian solvent. The total viscosity in the fluid $\eta = \eta_s + \eta_p$. The Giesekus model, derived from molecular ideas [38,39], has been successfully used to model experimental results, see, for example, [40–42]. Here we consider the case when $\zeta = \eta_s/\eta = 0$; when $\zeta = 0$ and $\alpha_m = 0$, (3.1) reduces to the upper-convected Maxwell fluid model [37].

We non-dimensionalise equations by scaling lengths by the squirmer radius a, velocities with the first squirming mode B_1, and stresses with $\eta B_1/a$, and obtain the dimensionless constitutive equation

$$\boldsymbol{\tau}^* + De\, \overset{\nabla}{\boldsymbol{\tau}}^* + \alpha_m De \boldsymbol{\tau}^* \cdot \boldsymbol{\tau}^* = \dot{\boldsymbol{\gamma}}^*, \tag{3.2}$$

where the Deborah number $De = \lambda B_1/a$. Henceforth, we drop the stars for convenience. We expand all flow quantities in a regular perturbation expansion in *De*, and using standard methods to calculate the flow fields in Stokes flow [33] obtain the swimming speed of the squirmer, up to $\mathcal{O}(De^3)$,

$$\begin{aligned} U = {} & \frac{2}{3} + \frac{2}{15}\beta(-1+\alpha_m)De \\ & + \frac{\beta^2(-20568 - 98136\alpha_m + 65266\alpha_m^2) + 84(-193 + 176\alpha_m(-3+2\alpha_m))}{45045} De^2 \\ & + \frac{\beta}{482431950}\Big(170\big(3005646 + \alpha_m(6190100 + 3\alpha_m \\ & \quad (-10014053 + 4815243\alpha_m)\big)\big) \\ & + \beta^2\big(224764987 + \alpha_m(1298121442 + 3\alpha_m \\ & \quad (-1659132865 + 875113652\alpha_m)\big)\big)\Big) De^3. \end{aligned} \tag{3.3}$$

At this point, examining Eq. (3.3) for specific values of β and α_m becomes instructive; we choose $\beta = -1$ for pushers, 0 for neutral squirmers, and 1 for puller type squirmers and $\alpha_m = 0.2$. These values correspond to the values used in the work of De Corato et al. [18]. From (3.3) we find, for pushers,

$$\frac{U}{U_N} = 1 + 0.16De - 2.05De^2 - 2.62De^3, \tag{3.4}$$

for pullers,

$$\frac{U}{U_N} = 1 - 0.16De - 2.05De^2 + 2.62De^3, \tag{3.5}$$

and for neutral squirmers,

$$\frac{U}{U_N} = 1 - 0.80De^2. \tag{3.6}$$

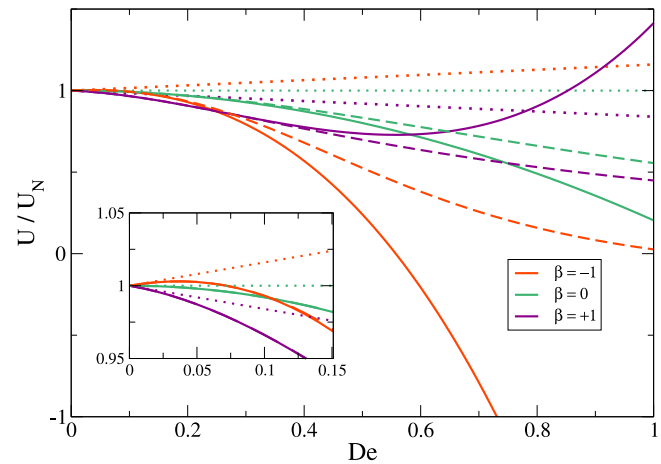

Fig. 1. Swimming speeds in the Giesekus fluid as a function of *De*. The solid lines include corrections up to $\mathcal{O}(De^3)$. The dashed lines are Padé approximations to the series for the speeds in the text. The dotted lines include only $\mathcal{O}(De)$ corrections. The addition of the higher order modes decreases the speeds of the squirmers. As seen here (dashed lines), all squirmers at large values of *De* swim slower than in a Newtonian fluid, as found in the numerical work of Zhu et al. [17].

The swimming speeds in (3.4), (3.5), and (3.6) are plotted in Fig. 1 along with their respective Padé approximant $P_2^1(De)$ [43]. When corrections up to only $\mathcal{O}(De)$ are considered, we note that pushers swim faster, pullers slower and neutral swimmers at the same speed as in a Newtonian fluid; this is shown in the work of De Corato et al. [18]. With Padé approximation of terms up to $\mathcal{O}(De^3)$, we note that all the squirmers swim slower than in a Newtonian fluid (except for very small values of *De*) as found in the numerical work of Zhu et al. [17] and De Corato et al. [18]. Clearly, the inclusion of higher order terms changes the theoretical predictions significantly.

One may calculate terms to even higher-order in the expansion to predict results for larger values of *De*. This is done by Housiadas and Tanner [44], up to $\mathcal{O}(De^8)$, for steady sedimentation of a passive sphere in a viscoelastic fluid. Housiadas and Tanner [44] also quantify when the results from the series should not be considered (using positive definiteness of the conformation tensor). Sauzade et al. [45] and Elfring and Lauga [5] also performed a higher-order perturbation analysis, using techniques to improve the convergence properties of the series, for the swimming speed of a two-dimensional swimming sheet where the small parameter was the amplitude of the waves on the sheet. We have not pursued these endeavours here, for the motivation for this study was to see the differences between the different viscoelastic models considering only the first few terms.

The results in the foregoing were obtained using the Giesekus model for viscoelasticity. They would remain qualitatively the same if one were to use the upper-convected Maxwell model. But what happens to a squirmer in a fluid of grade *n*, when the fluid is "regarded as a fluid in its own right, not necessarily an approximation to any other one" [28]?

3.2. A fluid of grade three

Consider an incompressible fluid of grade three [46]:

$$\boldsymbol{\tau} = \eta \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2 + \beta_1 \mathbf{A}_3 + \beta_2 [\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1] + \beta_3 (\mathrm{tr}\mathbf{A}_1^2)\mathbf{A}_1, \tag{3.7}$$

where η, α_1, α_2, β_1, β_2, and β_3 are material moduli. The equation is dimensional. For a list of works using fluids of grade three, see [47]. Thermodynamics stipulates [46] that

$$\begin{aligned} & \eta \geq 0 \quad \alpha_1 \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\eta\beta_3}, \\ & \beta_1 = 0 \quad \beta_2 = 0 \quad \beta_3 \geq 0. \end{aligned} \tag{3.8}$$

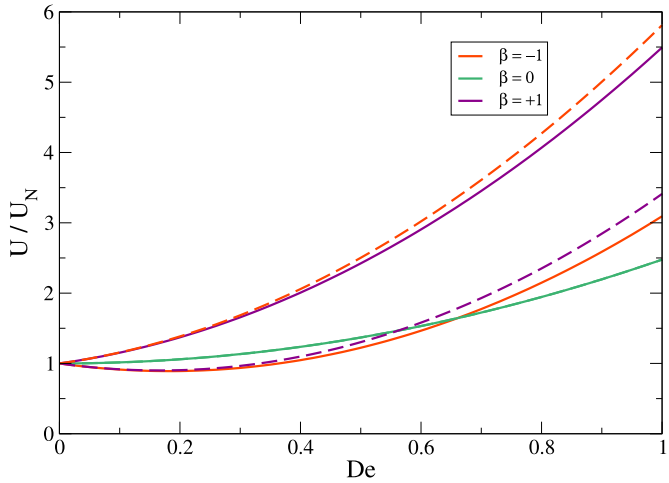

Fig. 2. Squirming speeds in fluids of grade three. Solid lines: $\mathcal{P} = 3/2$, $\mathcal{Q} = -7$. Dashed lines: $\mathcal{P} = 3/2$, $\mathcal{Q} = 5$. Solid and dashed lines for $\beta = 0$ overlap. Depending on the values of $\mathcal{Q}$ for a given $\mathcal{P}$, either of the puller or pusher can swim faster or slower than in a Newtonian fluid at small *De*.

We scale flow quantities as before, and consequently, Eqs. (3.7) with (3.8), in its dimensionless form, becomes

$$\boldsymbol{\tau} = \dot{\boldsymbol{\gamma}} + De\left[\overset{\Delta}{\dot{\boldsymbol{\gamma}}} + \mathcal{Q}\dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}}\right] + De^2[\mathrm{tr}(\dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}})\mathcal{P}\dot{\boldsymbol{\gamma}}]. \tag{3.9}$$

Here $De = \alpha_1 B_1/(\eta a)$, $\mathcal{Q} = \alpha_2/\alpha_1$ and $\mathcal{P} = \beta_3\eta/\alpha_1^2$ with $\mathcal{P} \geq 0$ and $|1 + \mathcal{Q}| \leq \sqrt{24\mathcal{P}}$. $\overset{\Delta}{\dot{\gamma}}$ is the lower convected derivative of $\dot{\gamma}$ [37], denoted by A_2 in Eq. (3.7). We expand all flow quantities in a regular perturbation expansion of *De* and calculate the propulsion speed up to $\mathcal{O}(De^2)$, which in dimensionless form comes to be

$$U = \frac{2}{3} - \frac{2}{15}\beta(1+\mathcal{Q})De - \frac{2\beta^2(1+\mathcal{Q})(161+559\mathcal{Q}) - 48\left(616+1383\beta^2\right)\mathcal{P}}{45045}De^2. \tag{3.10}$$

Note that when $\mathcal{P} = 0$, we obtain a fluid of grade two, where $1 + \mathcal{Q} = 0$ (2.7), and consequently, no contribution to the swimming speeds of the squirmers we consider. This is in contradiction to the results obtained through the weak *De* expansion in a Giesekus fluid to $\mathcal{O}(De)$ where pushers and pullers swim faster and slower, respectively, than in a Newtonian fluid. This was discussed in the exchange between Christov and Jordan [19] and De Corato et al. [20] described previously. It should be noted that the form of expression (3.10) up to $\mathcal{O}(De)$ is same as that of equation 31 in [18]. The difference in the coefficients is due to the different scale of velocities employed by the authors; we scale velocities as B_1, whereas De Corato et al. [18] use $2B_1/3$.

To observe the effects of a fluid of grade three, we choose $\mathcal{P} = 3/2$ (an arbitrary choice in as much as the physics of the problem is concerned). From Eq. (3.8) and its dimensionless form above, we know that $-7 \leq \mathcal{Q} \leq 5$. We plot the swimming speeds for two cases: $\mathcal{P} = 3/2$, $\mathcal{Q} = -7$ and $\mathcal{P} = 3/2$, $\mathcal{Q} = 5$ in Fig. 2.

From Fig. 2 and Eq. (3.10), we see that depending on the value of $\mathcal{Q}$, either of the puller or the pusher can swim faster than in a Newtonian fluid at $\mathcal{O}(De)$. The higher order correction, $\mathcal{O}(De^2)$, gives a positive contribution to the swimming speed.

In contrast to the results from the Giesekus fluid, the parameters in a fluid of grade three allow for a wider range of possibilities—either of pullers or pushers can swim faster or slower at small values of the Deborah number. Here, we demonstrate this using the parameter $\mathcal{Q}$ for a given $\mathcal{P}$. About this range of possibilities, perhaps it is useful to recall the observation from Truesdell [28] that "it is possible that two fluids of grade 3 could behave just alike in every viscometric test yet react altogether differently to some test of a different kind". At higher *De*, all squirmers swim faster in fluids of grade three than in a Newtonian fluid, when in Giesekus fluids they would swim slower.

4. Conclusion

We calculated the higher order corrections to the swimming speeds in two viscoelastic fluids—the Giesekus fluid and the fluid of grade three. The higher order corrections significantly add to the results at $\mathcal{O}(De)$; even at relatively small values of *De*, the corrections lead to qualitatively different speeds. This again raises the question about the range of values of *De* at which the expansion can accurately predict results (also see [23]). Importantly, we observe that the two fluids, the Giesekus fluid and the fluid of grade three, predict qualitatively different swimming speeds for the squirmers. Clearly, the answer to what viscoelastic model to use depends on what all we wish to model—in this, we are guided by experiments.

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References

[1] J. Wang, Can man-made nanomachines compete with nature biomotors? ACS Nano 3 (2009) 4–9, doi:10.1021/nn800829k.
[2] W. Gao, J. Wang, Synthetic micro/nanomotors in drug delivery, Nanoscale 6 (18) (2014) 10486–10494.
[3] B.J. Nelson, I.K. Kaliakatsos, J.J. Abbott, Microrobots for minimally invasive medicine, Annu. Rev. Biomed. Eng. 12 (1) (2010) 55–85, doi:10.1146/annurev-bioeng-010510-103409.
[4] J. Sznitman, P.E. Arratia, Locomotion Through Complex Fluids: An Experimental View, Springer, pp. 245–281. doi:10.1007/978-1-4939-2065-5_7
[5] G.J. Elfring, E. Lauga, Theory of Locomotion Through Complex Fluids, Springer, pp. 283–317. doi:10.1007/978-1-4939-2065-5_8
[6] P.A. Vasquez, M.G. Forest, Complex Fluids and Soft Structures in the Human Body, Springer, pp. 53–110. doi:10.1007/978-1-4939-2065-5_2
[7] G. Thurston, Viscoelasticity of human blood, Biophys. J. 12 (9) (1972) 1205–1217, doi:10.1016/S0006-3495(72)86156-3.
[8] S.K. Lai, Y.-Y. Wang, D. Wirtz, J. Hanes, Micro- and macrorheology of mucus, Adv. Drug Deliv. Rev. 61 (2009) 86–100, doi:10.1016/j.addr.2008.09.012.
[9] R.B. Bird, R.C. Armstrong, O. Hassager, Dynamics of Polymeric Liquids. Vol. 1: Fluid Mechanics, John Wiley, 1987.
[10] X.N. Shen, P.E. Arratia, Undulatory swimming in viscoelastic fluids, Phys. Rev. Lett. 106 (2011) 208101, doi:10.1103/PhysRevLett.106.208101.
[11] B. Qin, A. Gopinath, J. Yang, J.P. Gollub, P.E. Arratia, Flagellar kinematics and swimming of algal cells in viscoelastic fluids, Sci. Rep. 5 (2015).
[12] C. Li, B. Qin, A. Gopinath, P.E. Arratia, B. Thomases, R.D. Guy, Flagellar swimming in viscoelastic fluids: role of fluid elastic stress revealed by simulations based on experimental data, J. R. Soc. Interface 14 (135) (2017) 20170289, doi:10.1098/rsif.2017.0289.
[13] E. Lauga, Propulsion in a viscoelastic fluid, Phys. Fluids 19 (2007) 083104, doi:10.1063/1.2751388.
[14] M.J. Lighthill, On the squirming motion of nearly spherical deformable bodies through liquids at very small Reynolds numbers, Commun. Pure Appl. Math. 5 (1952) 109–118, doi:10.1002/cpa.3160050201.
[15] J.R. Blake, A spherical envelope approach to ciliary propulsion, J. Fluid Mech. 46 (1971) 199–208, doi:10.1017/S002211207100048X.
[16] T.J. Pedley, Spherical squirmers: models for swimming micro-organisms, IMA J. Appl. Math. 81 (2016) 488–521, doi:10.1093/imamat/hxw030.
[17] L. Zhu, E. Lauga, L. Brandt, Self-propulsion in viscoelastic fluids: Pushers vs. pullers, Phys. Fluids 24 (2012) 051902, doi:10.1063/1.4718446.
[18] M. De Corato, F. Greco, P.L. Maffettone, Locomotion of a microorganism in weakly viscoelastic liquids, Phys. Rev. E 92 (2015) 053008, doi:10.1103/PhysRevE.92.053008.
[19] I.C. Christov, P.M. Jordan, Comment on "Locomotion of a microorganism in weakly viscoelastic liquids", Phys. Rev. E 94 (2016) 057101, doi:10.1103/PhysRevE.94.057101.
[20] M. De Corato, F. Greco, P.L. Maffettone, Reply to "Comment on 'Locomotion of a microorganism in weakly viscoelastic liquids' ", Phys. Rev. E 94 (2016) 057102, doi:10.1103/PhysRevE.94.057102.

[21] T. Ishikawa, M.P. Simmonds, T.J. Pedley, Hydrodynamic interaction of two swimming model micro-organisms, J. Fluid Mech. 568 (2006) 119–160, doi:10.1017/S0022112006002631.
[22] C. Datt, L. Zhu, G.J. Elfring, O.S. Pak, Squirming through shear-thinning fluids, J. Fluid Mech. 784 (2015) R1, doi:10.1017/jfm.2015.600.
[23] C. Datt, G. Natale, S.G. Hatzikiriakos, G.J. Elfring, An active particle in a complex fluid, J. Fluid Mech. 823 (2017) 675–688, doi:10.1017/jfm.2017.353.
[24] O.S. Pak, E. Lauga, Generalized squirming motion of a sphere, J. Eng. Math. 88 (2014) 1–28, doi:10.1007/s10665-014-9690-9.
[25] J. Dunn, K. Rajagopal, Fluids of differential type: critical review and thermodynamic analysis, Int. J. Eng. Sci. 33 (5) (1995) 689–729, doi:10.1016/0020-7225(94)00078-X.
[26] R.I. Tanner, Engineering Rheology, vol. 52, OUP Oxford, 2000.
[27] B.D. Coleman, W. Noll, An approximation theorem for functionals, with applications in continuum mechanics, Arch. Ration. Mech. Anal. 6 (1) (1960) 355–370, doi:10.1007/bf00276168.
[28] C. Truesdell, The meaning of viscometry in fluid dynamics, Annu. Rev. Fluid Mech. 6 (1) (1974) 111–146, doi:10.1146/annurev.fl.06.010174.000551.
[29] O.S. Pak, L. Zhu, L. Brandt, E. Lauga, Micropropulsion and microrheology in complex fluids via symmetry breaking, Phys. Fluids 24 (2012) 103102, doi:10.1063/1.4758811.
[30] A.M. Ardekani, R.H. Rangel, D.D. Joseph, Two spheres in a free stream of a second-order fluid, Phys. Fluids 20 (6) (2008) 063101, doi:10.1063/1.2917976.
[31] P. Brunn, The motion of rigid particles in viscoelastic fluids, J. Non-Newton. Fluid Mech. 7 (1980) 271–288, doi:10.1016/0377-0257(82)80019-0.
[32] J.E. Dunn, R.L. Fosdick, Thermodynamics, stability, and boundedness of fluids of complexity 2 and fluids of second grade, Arch. Ration. Mech. Anal. 56 (3) (1974) 191–252, doi:10.1007/BF00280970.
[33] J. Happel, H. Brenner, Low Reynolds Number Hydrodynamics, Martinus Nijhoff, 1983.
[34] E. Lauga, Locomotion in complex fluids: Integral theorems, Phys. Fluids 26 (2014) 081902, doi:10.1063/1.4891969.
[35] E. Lauga, Life at high Deborah number, Europhys. Lett. 86 (2009) 64001, doi:10.1209/0295-5075/86/64001.
[36] R. Poole, The Deborah and Weissenberg numbers, Rheol. Bull. The British Society of Rheology 53 (2012) 32–39.
[37] A. Morozov, S.E. Spagnolie, Introduction to Complex Fluids, Springer, pp. 3–52.
[38] H. Giesekus, A simple constitutive equation for polymer fluids based on the concept of deformation-dependent tensorial mobility, J. Non-Newton. Fluid Mech. 11 (1) (1982) 69–109, doi:10.1016/0377-0257(82)85016-7.
[39] R.B. Bird, J.M. Wiest, Constitutive equations for polymeric liquids, Annu. Rev. Fluid Mech. 27 (1) (1995) 169–193, doi:10.1146/annurev.fl.27.010195.001125.
[40] G. D'Avino, M.A. Hulsen, F. Snijkers, J. Vermant, F. Greco, P.L. Maffettone, Rotation of a sphere in a viscoelastic liquid subjected to shear flow. Part i: simulation results, J. Rheol. 52 (6) (2008) 1331–1346, doi:10.1122/1.2998219.
[41] F. Snijkers, G. DAvino, P.L. Maffettone, F. Greco, M. Hulsen, J. Vermant, Rotation of a sphere in a viscoelastic liquid subjected to shear flow. Part ii. Experimental results, J. Rheol. 53 (2) (2009) 459–480, doi:10.1122/1.3073052.
[42] F. Snijkers, G. DAvino, P. Maffettone, F. Greco, M. Hulsen, J. Vermant, Effect of viscoelasticity on the rotation of a sphere in shear flow, J. Non Newton. Fluid Mech. 166 (7) (2011) 363–372, doi:10.1016/j.jnnfm.2011.01.004.
[43] C.M. Bender, S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory, Springer Science & Business Media, 2013.
[44] K.D. Housiadas, R.I. Tanner, A high-order perturbation solution for the steady sedimentation of a sphere in a viscoelastic fluid, J. Non-Newton. Fluid Mech. 233 (2016) 166–180, doi:10.1016/j.jnnfm.2016.03.014. Papers presented at the Rheology Symposium in honor of Prof. R. I. Tanner on the occasion of his 82nd birthday, in Vathi, Samos, Greece
[45] M. Sauzade, G.J. Elfring, E. Lauga, Taylor's swimming sheet: analysis and improvement of the perturbation series, Phys. D 240 (20) (2011) 1567–1573, doi:10.1016/j.physd.2011.06.023.
[46] R.L. Fosdick, K.R. Rajagopal, P. Chadwick, Thermodynamics and stability of fluids of third grade, Proc. R. Soc. Lond. A. 369 (1738) (1980) 351–377, doi:10.1098/rspa.1980.0005.
[47] R.B. Bird, Useful non-Newtonian models, Annu. Rev. Fluid Mech. 8 (1) (1976) 13–34, doi:10.1146/annurev.fl.08.010176.000305.